

INTERSECTIONS OF DIAGONAL ORBITS

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ABSTRACT. Let $A \subseteq SL_n(\mathbb{R})$ group of diagonal matrices with positive diagonal, let $ST_n \subseteq X_n := SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ be the set of stable lattices, and let $WR_n \subseteq X_n$ be the set of well-rounded lattices. We prove that any A -orbit in X_n intersects both ST_n and WR_n .

1. INTRODUCTION

Let $A \subseteq SL_n(\mathbb{R})$ be the diagonal subgroup and let $X_n := SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ be the space of lattices. It is believed that Minkowski suggested the following conjecture:

Conjecture 1.1. *For every $\Lambda \in X_n$ and $p \in \mathbb{R}^n$ there exist $a \in A$ and $v \in \Lambda$ such that $\|a(p - v)\| \leq \frac{\sqrt{n}}{2}$.*

The conjecture was proved for $n \leq 9$. The first proofs for $n \leq 5$ used the following strategy, known as the Remak-Davenport approach. Define the set of well-rounded lattices $WR_n \subseteq X_n$ as the set of all lattices such that all the Minkowski successive minima are equal. The Remak-Davenport approach states that to prove Minkowski's conjecture it is enough to prove the following two statements.

- (W_n) For every lattice $\Lambda \in X_n$ we have $A\Lambda \cap WR_n \neq \emptyset$.
- (C_n) For every $\Lambda \in WR_n$,

$$\sup_{p \in \mathbb{R}^n} \inf_{v \in \Lambda} \|p - v\| \leq \frac{\sqrt{n}}{2}.$$

The cases $n = 2, 3, 4, 5$ were proven by Minkowski [12], Remak [14], Dyson [3], and Skubenko [16], respectively.

McMullen [10] proved a weaker version of (W_n), that, combined with a result of Birch and Swinnerton-Dyer [1], demonstrated that if $(C_1), (C_2), \dots, (C_n)$ holds then Minkowski's Conjecture holds for n . Woods [17] proved (C_n) for $n = 6$, and in [5], [6], and [8] Hans-Gill, Kathuria, Raka, and Sehmi proved (C_n) for $n = 7, 8, 9$. In particular, the Minkowski Conjecture indeed holds for $n \leq 9$.

Regev, Shapira, and Weiss [13] proved that (C_n) is false for $n \geq 30$, and therefore the Remak-Davenport approach is bound to fail. Shapira and Weiss [15] suggested a similar approach replacing the set of well-rounded lattices by the set of stable lattices (see Definition 2.1 below).

As for (W_n) , McMullen [10] proved that any bounded orbit closure $\overline{A\Lambda} \subseteq X$ intersects WR_n . Levin, Shapira, and Weiss [9] proved that every closed orbit $A\Lambda \subseteq X_n$ intersects WR_n . Shapira and Weiss [15] proved that every orbit closure $\overline{A\Lambda} \subseteq X$ intersects the set of stable lattices ST_n , and concluded that the analog of (C_n) , when replacing WR_n by ST_n , implies Minkowski's Conjecture. In this paper we prove the following result, which strengthens results in [10], [9], and [15].

Theorem 1.2. *For every $\Lambda \in X_n$ the orbit $A\Lambda$ intersects ST_n and WR_n w.r.t. any norm.*

The proof is inspired by [10] and is a combination of a topological claim and some lattice geometry. To state the topological theorem, we need the concept of invariance dimension. Recall that \mathbb{R}^n acts on its subsets by translations.

Definition 1.3. The *invariance dimension* of a convex open set $U \subseteq \mathbb{R}^n$ is the dimension of its stabilizer over \mathbb{R}^n , that is,

$$\text{invdim } U := \dim \text{stab}_{\mathbb{R}^n}(U).$$

By convention $\text{invdim } \emptyset := -\infty$.

The topological result that we need and that extends theorem 5.1 in [10] is the following.

Theorem 1.4. *Let \mathfrak{U} be an open cover of \mathbb{R}^n . Assume that*

(1) *the cover*

$$\{\text{conv} U : U \in \mathfrak{U}\}$$

*is locally finite;*¹

(2) *for every $k \leq n$ and k different sets $U_1, \dots, U_k \in \mathfrak{U}$ one has*

$$\text{invdim } \text{conv}(U_1 \cap U_2 \cap \dots \cap U_k) \leq n - k.$$

Then there are $n + 1$ sets in \mathfrak{U} with nontrivial intersection.

¹An open cover is *locally finite* if every compact set intersects only finitely many cover elements.

2. PROOF OF THEOREM 1.2

We will provide some notations, most are taken from [10]. Define the *Minkowski successive minima* of a lattice Λ by

$$\lambda_i(\Lambda) := \inf\{r > 0 : \dim \text{span}\{v \in \Lambda : |v| < r\} \geq i\}.$$

Let $\text{WR}_n \subseteq X_n$ be the set of all lattices for which all Minkowski successive minima are equal. Although the standard definition of WR_n uses the euclidean norm $|\cdot|$, here we consider the analogous definition with an arbitrary fixed norm.

The Harder-Narasimhan filtration was defined in [7] and described nicely by Grayson [4]. Its construction for standard lattices in \mathbb{R}^n goes as follows. For every discrete subgroup $\Gamma < \mathbb{R}^n$, denote by $\text{covol } \Gamma$ the Euclidean volume of the group $\text{span } \Gamma / \Gamma$. By convention $\text{covol } \{0\} := 1$. We associate to Γ the point $p_\Gamma := (\text{rank}(\Gamma), \log \text{covol } \Gamma) \in \mathbb{R}^2$.

For every lattice $\Lambda \in X_n$ define $S_\Lambda := \{p_\Gamma : \Gamma \leq \Lambda\}$. Denote the extreme points of $\text{conv}(S_\Lambda)$ by p_0, \dots, p_k , and, for each $0 \leq i \leq k$, let $\Gamma_i \leq \Gamma$ satisfy $p_i = p_{\Gamma_i}$. A result of the Harder-Narasimhan filtration states that up to reordering, $\{0\} = \Gamma_0 < \dots < \Gamma_k = \Lambda$, are of strictly increasing ranks. Furthermore, if $p(\Gamma)$ is an extreme point, then Γ is the unique subgroup that is associated to this point. In addition, for every $0 \leq i \leq k$ one has $\text{covol } \Gamma_i \leq 1$. The filtration $\{0\} = \Gamma_0 < \dots < \Gamma_k = \Lambda$ is called the Harder-Narasimhan Filtration.

Definition 2.1. The set of *stable lattices* ST_n is the set of all lattices $\Lambda \in X_n$ such that the Harder-Narasimhan filtration of Λ contains only $\{0\}$ and Λ , that is, for every $\Gamma \leq \Lambda$ one has $\text{covol } \Gamma \geq 1$.

Wedge product geometry. Denote $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard basis of \mathbb{R}^n . A basis for $\bigwedge^k \mathbb{R}^n$ is given by $\mathbf{e}_J := \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_k}$ for $J = \{0 < j_1 < \dots < j_k \leq n\}$. Its dual basis is denoted $\{\varphi_J : \#J = k\} \subseteq \left(\bigwedge^k \mathbb{R}^n\right)^*$.

A vector in the k 'th wedge product is called a k -vector. For simplicity, for every k -vector $v \in \bigwedge^k \mathbb{R}^n$ we use the norm

$$\|\omega\|_{k\text{-vec}} := \max_J |\varphi_J(\omega)|$$

and

$$\text{supp } \omega := \{J : \varphi_J(\omega) \neq 0\}.$$

Do not confuse the arbitrary norm $|\cdot|$ of \mathbb{R}^n with $\|\cdot\|_{1\text{-vec}}$ on $\bigwedge^1 \mathbb{R}^n \cong \mathbb{R}^n$.

Measured subspaces. A k -dimensional *measured subspace* is a real vector subspace $M \subseteq \mathbb{R}^n$ equipped with a nonzero k -vector $\det(M) \in \bigwedge^k M$, chosen up to sign. We denote the set of k dimensional measured subspaces by $\mathfrak{G}_{n,k}$. For every k dimensional measured subspace M we define $\|M\|_{MS} := \|\det M\|_{k\text{-vec}}$.

Any discrete subgroup $\Gamma < \mathbb{R}^n$ gives rise to a measured space $M(\Gamma) \in \mathfrak{G}_{n, \text{rank } \Gamma}$; the space is $\text{span } \Gamma$, and $\det M(\Gamma) = v_1 \wedge \dots \wedge v_k$ for a basis v_1, \dots, v_k of Γ .

For a vector space V we define its support to be $\text{supp}(v_1 \wedge \dots \wedge v_k)$ for some (any) basis v_i of V , this is well-defined because changing the basis only multiplies $v_1 \wedge \dots \wedge v_k$ by a nonzero scalar.

An alternative definition of $\text{supp } v$ is

$$\text{supp } v := \{J \subseteq \{1, \dots, n\} \text{ of size } k : \pi_J|_v \text{ is injective}\},$$

where $\pi_J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the projection setting all coordinates not in J to 0.

Flags.

The main object that we use in the proof is the concept of a measured flag. A *measured flag* is a sequence of measured spaces $\{0 = v_0 < v_1 < \dots < v_l = \mathbb{R}^d\}$. We impose no restrictions on the volume elements. Denote the set of measured flag by \mathfrak{F}_n and for every measured flag $F = \{0 = v_0 < v_1 < \dots < v_l = \mathbb{R}^d\}$ define $\|F\|_F := \max_{l>0} \|v_l\|_{MS}$. We will investigate functions $F : A \rightarrow \mathfrak{F}_n$ with the following properties.

Definition 2.2. A function $F : A \rightarrow \mathfrak{F}_n$ is *bounded* if

$$\sup_{a \in A} \|F(a)\|_F < \infty.$$

It is *lower locally invariant* if for every $a \in A$ there is a neighborhood $U \subseteq A$ of the identity matrix such that $a'F(a) \subseteq F(a'a)$ for every $a' \in U$.

F is *discrete* if the set

$$\{a^{-1} \det v : a \in A, v \in F(a)\}$$

is discrete in $\bigsqcup_{k=0}^n \bigwedge^k \mathbb{R}^n$.

Theorem 2.3. *For any discrete bounded lower locally invariant F there is a point $a \in A$ such that $F(a)$ is the trivial flag $\{0 < \mathbb{R}^n\}$.*

Proof of Theorem 1.2 using Theorem 2.3. Fix a lattice Λ_0 and let $B(r) \subseteq \mathbb{R}^n$ be the ball of radius r with respect to the norm $|\cdot|$. For every lattice $\Lambda \in X_n$ define the Minkowski measured flag by

$$F_{\text{Mink}}(\Lambda) := \{\text{span } B(r) \cap \Lambda : r > 0\}$$

and for every $v \in F(\Lambda)$ the volume element is given by $M(v \cap \Lambda)$. By Minkowski's second theorem one can see that there is a constant C_n depending only on n such that $\|F(\Lambda)\|_F \leq C_n$.

We will prove that for some $a \in A$ we have $a\Lambda_0 \in \text{WR}_n$. We apply Theorem 2.3 to the flag

$$F(a) := F_{\text{Mink}}(a\Lambda_0).$$

By the previous discussion this flag is bounded. It is discrete since $\bigsqcup \wedge^k \Lambda_0$ is discrete. It is lower locally invariant by the definition of F . The result follows since $\text{WR}_n = \{\Lambda \in X_n : F(\Lambda) = \{0 < \mathbb{R}^n\}\}$. A similar proof, using the Harder-Narasimhan filtration instead of the Minkowski measured flag, shows that ST_n intersect every A orbit. \square

The rest of this section is dedicated to the proof of Theorem 2.3 Using Theorem 1.4.

Denote $[a] := \{1, \dots, a\}$. We will prove the following simple observation.

Lemma 2.4. *For every flag $F = \{0 = v_0 < v_1 < \dots < v_l = \mathbb{R}^n\}$ there exist a permutation σ of $[n]$ such that $\sigma([\dim v_i]) \in \text{supp } v_i$ for every $0 \leq i \leq l$.*

Proof. Without loss of generality add some subspaces to the flag and assume that $l = n$, that is, all dimensions appear in F and $\dim v_i = i$ for every $0 \leq i \leq n$. Recall that for every $J \subseteq [n]$ we denoted by π_J the projection $\mathbb{R}^n \rightarrow \mathbb{R}^n$ setting all coordinates not in J to 0, which has rank $\#J$.

We construct the permutation σ inductively. At the k 'th stage we will construct $\sigma(k)$ such that $\pi_{\sigma([k])}|_{v_k}$ is a bijection (for $k = 0$ this assumption is vacuous). Suppose by induction for some $J = \sigma([k])$ we have that $\pi_J|_{v_k}$ is a bijection. We will show that there exist $j' \notin J$ such that $\pi_{J \cup \{j'\}}|_{v_{k+1}}$ is a bijection and define $\sigma(k+1) = j'$. Since $\dim v_{k+1} > k$ there is a nontrivial vector $v \in \ker \pi_J|_{v_{k+1}}$. Since $v \in \ker \pi_J$ all its J coordinates vanish. Since it is nontrivial, there is j' such that the j' coordinate of v is nontrivial. Denote $J' := J \cup \{j'\}$. Since the j' coordinate of v is nontrivial, $\pi_{J'}(v) \neq 0$. But $\pi_J(v) = \pi_J \circ \pi_{J'}(v) = 0$ and hence $k = \dim \pi_J(v_{k+1}) < \dim \pi_{J'}(v_{k+1})$. Therefore $\pi_{J'}|_{v_{k+1}}$ is a bijection, as desired. \square

Convex sets

Lemma 2.5. *If $\emptyset \neq U_1 \subseteq U_2 \subseteq \mathbb{R}^n$ are open convex sets then $\text{invdim } U_1 \leq \text{invdim } U_2$.*

Proof. Assume without loss of generality that $0 \in U_1$. Since for every open convex set U that contains 0 we have

$$\text{stab}_{\mathbb{R}^n} U = \{v \in \mathbb{R}^n : \mathbb{R}v \subseteq U\},$$

the result follows. \square

Define

$$\begin{aligned} \exp : \mathbb{R}_0^{n-1} &:= \left\{ (x_1, \dots, x_n) : \sum_{i=1}^n x_i = 0 \right\} \rightarrow A \\ (x_1, \dots, x_n) &\mapsto \text{diag}(\exp x_1, \dots, \exp x_n), \end{aligned}$$

and $\log : A \rightarrow \mathbb{R}_0^{n-1}$ be the inverse function. We will identify A and \mathbb{R}_0^{n-1} using this transformation and push all the notions of convexity that are defined on \mathbb{R}_0^{n-1} to A .

Since the exponential function $x \mapsto e^x$ is convex, and since maximum preserves convexity, the function $a \mapsto \|aM\|_{MS}$ is a convex function for all $M \in \mathfrak{G}_{n,k}$ and so is $a \mapsto \|aF\|_F$ for all $F \in \mathfrak{F}_n$.

Proof of Theorem 2.3 using Theorem 1.4. Assume to the contrary that $F : A \rightarrow \mathfrak{F}_n$ is discrete, lower locally invariant, nowhere trivial, and bounded by $c_F > 0$. Construct the following cover of \mathbb{R}_0^{n-1} . For every $0 < k < n$ and k -dimensional measured space v define $U_v := \{a \in A : av \in F(a)\}$. Let \mathfrak{U} be the collection of sets $\{U_v\}$, where v ranges over all k -dimensional measured spaces with $0 < k < n$. Since F is nowhere trivial, we deduce that \mathfrak{U} is a cover of A .

To use Theorem 1.4 we need to prove that its Conditions (1) and (2) holds. To prove that Condition (1) holds, let \mathfrak{U}' be the collection of sets $U'_v := \{a \in A : \|av\|_{MS} \leq c_F\}$. Since F is bounded by c_F , we have $U'_v \supseteq U_v$ for every measured space v . Consequently, \mathfrak{U}' is a cover, and since F is discrete, it is locally finite. Hence \mathfrak{U} is locally finite as well.

To prove Condition (2) we will classify intersection of elements in \mathfrak{U} . Let $U_{v_1}, U_{v_2}, \dots, U_{v_l}$ be elements of \mathfrak{U} that have a nontrivial intersection $V \neq \emptyset$. For all $a \in V$ we have $av_1, \dots, av_l \in F(a)$, and hence v_1, \dots, v_l form a flag. Assume without loss of generality that $0 < v_1 < v_2 < \dots < v_l < \mathbb{R}^n$. By Lemma 2.4 there exists a permutation $\sigma : [n] \rightarrow [n]$ such that $\sigma([\dim v_k]) \in \text{supp } v_k$. Assume without loss of generality that σ is the identity permutation. Note that for all $1 \leq k \leq l$, $\vec{x} \in \mathbb{R}_0^{n-1}$ one has

$$\varphi_{[\dim v_k]}(\exp(\vec{x})v_k) = \exp(\psi_{\dim v_k} \vec{x})\varphi_{[\dim v_k]}(v_k),$$

where

$$\begin{aligned} \psi_m : \mathbb{R}_0^{n-1} &\rightarrow \mathbb{R}, \\ \vec{x} = (x_1, \dots, x_n) &\mapsto x_1 + \dots + x_m. \end{aligned}$$

Denote $c_k := |\varphi_{[\dim v_k]}(v_k)|$. For every $\vec{x} \in \log V$ one has

$$c_F > \|F(\exp \vec{x})\|_F \geq \max_{k=1}^l \|\exp(\vec{x})v_k\|_{MS} \geq \max_{k=1}^l \exp(\psi_{\dim v_k} \vec{x})c_k,$$

and hence the set $\log V$ is contained in $P := \bigcap_{k=1}^l \psi_k^{-1}(-\infty, \log c_F - \log c_k)$. Since the functionals ψ_k are linearly independent, the set P satisfies $\text{invdim } P = n - 1 - l$, and hence $\text{invdim conv}(V) \leq n - 1 - l$.

We proved that the conditions of Theorem 1.4 holds, and therefore the conclusion is as well: there is a nontrivial intersection of n sets of \mathfrak{U} . As shown

above, this intersection corresponds to a nontrivial flag with n nontrivial elements, which is a contradiction. \square

3. PROOF OF THEOREM 1.4

3.1. Sketch of proof. The proof of Theorem 1.4 is a modified version of the proof of Theorem 5.1 in [10]. The main steps of the two proofs are the following:

- (1) We construct a complex of presheaves

$$\mathcal{F} : 0 \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \dots$$

on \mathbb{R}^n such that the n 'th cohomology of \mathbb{R}^n w.r.t. \mathcal{F} , denoted $H_{\mathcal{F}}^n(\mathbb{R}^n)$, is nontrivial. We select a family \mathfrak{E} of open subsets of \mathbb{R}^n and calculate their \mathcal{F} -cohomologies.

- (2) Using conditions (1) and (2) we construct for every set of the form $V := U_1 \cap U_2 \cap \dots \cap U_k$ a nice set $V \subseteq E(V) \in \mathfrak{E}$ for which the $(n - k)$ \mathcal{F} -cohomology is trivial, and such that whenever $V_1 \subseteq V_2$ we have $E(V_1) \subseteq E(V_2)$.
- (3) We complete the proof using some cohomological algebra. We construct a Čech-deRham double complex \mathcal{A} using \mathcal{F} and \mathfrak{U} . We prove exactness in the Čech direction, and conclude that the \mathcal{F} -cohomology of \mathbb{R}^n is equal to the total cohomology of \mathcal{A} . We cover \mathcal{A} by a double complex \mathcal{B} , built with E instead of the intersections themselves. We show that the restriction map $\mathcal{B} \rightarrow \mathcal{A}$ is onto on the cohomologies. Then we show that \mathcal{B} is exact in the \mathcal{F} direction on the n 'th level, and hence any element in the n cohomology class of \mathcal{B} can be represented in the class that represents E of intersection of $n + 1$ elements. Since there is a nontrivial n dimensional \mathcal{F} cohomology class in \mathbb{R}^n there is a nonempty intersection of $n + 1$ elements of U .

Since \mathcal{F} is not a sheaf, some work is needed to achieve exactness.

The differences between the proof of Theorem 1.4 and of McMullen are the following:

- McMullen uses the complex of bounded forms while we use the complex of boundedly supported forms.
- For the family \mathfrak{E} McMullen uses cylinders, while we use convex sets.
- The cohomology calculation is different: McMullen calculates it directly while we use the Mayer-Vietoris sequence.
- The Čech-deRham double complex is different: McMullen used direct sum of normed spaces while we use standard direct sum.

3.2. Boundedly supported forms. Denote by $B(r) \subseteq \mathbb{R}^n$ the open ball of radius r around 0. For every open set $U \subseteq \mathbb{R}^n$ denote by $\Omega^k(U)$ the set of k -forms on U and by $\Omega_{bs}^k(U)$ the set of k -forms on U that vanish outside $B(r)$ for some $r > 0$. Recall the differential transformation $d = d_k : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$. Denote by $H^*(U)$ the $\Omega^*(U)$ -cohomology group and by $H_{bs}^*(U)$ the $\Omega_{bs}^*(U)$ -cohomology group.

Definition 3.1. For every convex open set $U \subseteq \mathbb{R}^n$ we define $\deg U$ as follows. If the projection of U to $\mathbb{R}^n / \text{stab}_{\mathbb{R}^n}(U)$ is bounded then $\deg U := \text{invdim}(U)$; otherwise, $\deg U := -\infty$.

For example, the convex region $U_0 \subseteq \mathbb{R}^2$ bounded by a parabola satisfies $\text{invdim } U_0 = 0$ and $\deg U_0 = -\infty$, and the open cylindrical neighborhood of a line $U_1 \subseteq \mathbb{R}^3$ satisfies $\text{invdim } U_1 = \deg U_1 = 1$.

Lemma 3.2. *If $U \subseteq \mathbb{R}^n$ is an unbounded open convex set and $\text{invdim } U = 0$, then there is a functional φ such that*

$$\{x \in U : \varphi(x) < r\} \text{ is bounded for every } r > 0. \quad (3.1)$$

Proof. Assume without loss of generality that $0 \in U$. Denote by

$$A = A(U) := \{x \in \mathbb{R}^n : \forall \lambda > 0, \lambda x \in U\}$$

the union of all rays from 0 that are contained in U . Note that $A = \bigcap_{\lambda > 0} \lambda U$ is the intersection of convex sets and hence convex. Since $^2\frac{1}{2}\bar{U} \subseteq U$ we have $A = \bigcap_{\lambda > 0} \lambda \bar{U}$, and hence A is closed. Let S^{n-1} be the $n-1$ unit sphere and denote $C = C(U) := S^{n-1} \cap A$. Since

$$C = \bigcap_{\lambda > 0} (S^{n-1} \cap \lambda \bar{U})$$

is the intersection of nonempty compact sets that decrease as λ goes to 0, it is nonempty. We argue that $0 \notin \text{conv}(C)$. Indeed if $0 \in \text{conv}(C)$ then there exist $l > 0$, $v_1, \dots, v_l \in C$ and positive $\alpha_1, \dots, \alpha_l$ such that $\sum_{i=1}^l \alpha_i v_i = 0$. Since U is convex it follows that $V := \text{span}\{v_1, \dots, v_l\} \subseteq U$, and hence U is invariant to translations by vectors in V , which contradicts the assumption that $\text{invdim } U = 0$. Hence, $0 \notin \text{conv } C$, and there exists a functional $\varphi \in (\mathbb{R}^n)^*$ such that $\varphi|_C > 1$. We will show that φ satisfies Equation (3.1). Otherwise, there exists $r > 0$ such that the set $U' := \{x \in U : \varphi(x) < r\}$ is unbounded. In particular

$$\emptyset \neq C(U') \subseteq C(U) = C. \quad (3.2)$$

² \bar{U} is the closure of U .

On the other hand

$$C(U') \subseteq A(U') = \bigcap_{\lambda > 0} \lambda U' \subseteq \bigcap_{\lambda > 0} \{x \in U : \varphi(x) < r\} = \{x \in U : \varphi(x) \leq 0\},$$

which, together with Equation (3.2), contradicts $\varphi|_C > 1$. Therefore U' is bounded, as desired. \square

Theorem 3.3. *For every convex open set $U \subseteq \mathbb{R}^n$ and every $k \geq 0$ we have*

$$H_{bs}^k(U) \cong \begin{cases} \mathbb{R} & k = \deg U, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove the claim by induction on $\text{invdim } U$. Assume first that $\text{invdim } U = 0$. If U is bounded, we have $\Omega_{bs}^*(U) = \Omega^*(U)$, and the claim holds since U is convex. Assume now that U is unbounded and define $\Omega_o^k(U) := \Omega^k(U)/\Omega_{bs}^k(U)$. Let φ be functional satisfying Equation (3.1). Choose $\omega \in \Omega^k(U)$ that represents a cocycle in $H_o^k(U)$, the Ω_o^k -cohomology of U . Then there is $r > 0$ such that $d\omega$ vanishes on $V := U \cap \varphi^{-1}(r, \infty)$. Since V is a nonempty convex set, $H^k(V) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases}$, and hence there exists $\varpi \in \Omega^{k-1}(V)$ such that

$$\omega = \begin{cases} \text{const} & \text{if } k = 0 \\ d\varpi & \text{otherwise} \end{cases} \text{ in } V.$$

One can find a $(k-1)$ -form $\varpi' \in \Omega^{k-1}(U)$ that agrees with ϖ on $U \cap \varphi^{-1}(r+1, \infty)$, and thus $[\omega] \in H_o^k(U)$ is either trivial, or equivalent to the constant function if $k = 0$. One can see that $H_o^0(U) \cong \mathbb{R}$, since the constant functions in $\Omega_o^0(U)$ generate a nontrivial class. By definition the following is a short exact sequence of complexes:

$$0 \longrightarrow \Omega_{bs}^*(U) \longrightarrow \Omega^*(U) \longrightarrow \Omega_o^*(U) \longrightarrow 0.$$

By the snake lemma the following is a long exact sequence of cohomologies:

$$\begin{aligned} 0 \rightarrow H_{bs}^0(U) \rightarrow H^0(U) \rightarrow H_o^0(U) \rightarrow \\ H_{bs}^1(U) \rightarrow H^1(U) \rightarrow H_o^1(U) \rightarrow \dots \end{aligned}$$

Note that the arrow $H^k(U) \rightarrow H_o^k(U)$ is an isomorphism for every k . For $k = 0$ the two groups are isomorphic to \mathbb{R} and the arrow is a monomorphism. For $k > 0$ both are trivial. Therefore, all the cohomologies in the sequence $H_{bs}^*(U)$ are 0, and the proof for the case $\text{invdim } U = 0$ is complete.

For the induction step, suppose $\text{invdim } U = k > 0$. Assume without loss of generality that $U = \mathbb{R}^k \times U'$ for $U' \subseteq \mathbb{R}^{n-k}$ with $\text{invdim } U' = 0$. Write $U = U_1 \cup U_2$ where $U_1 := U \cap \{x_1 \geq -1\}$ and $U_2 := U \cap \{x_1 \leq 1\}$. Denote

$V := U_1 \cap U_2$. Note that $\text{invdim } U_1 = \text{invdim } U_2 = \text{invdim } V = k - 1$, $\deg U_1 = \deg U_2 = -\infty$, and $\deg V = \deg U - 1$. Note that

$$0 \longrightarrow \Omega_{bs}^*(U) \xrightarrow{\alpha \mapsto (\alpha, \alpha)} \Omega^*(U_1) \times \Omega^*(U_2) \xrightarrow{(\alpha, \beta) \mapsto \alpha - \beta} \Omega_o^*(V) \longrightarrow 0$$

is a short exact sequence of complexes and by the snake lemma there is a long exact sequence of cohomologies

$$\begin{aligned} 0 \rightarrow H_{bs}^0(U) \rightarrow H_{bs}^0(U_1) \oplus H_{bs}^0(U_2) \rightarrow H_{bs}^0(V) \rightarrow \\ H_{bs}^1(U) \rightarrow H_{bs}^1(U_1) \oplus H_{bs}^1(U_2) \rightarrow H_{bs}^1(V) \rightarrow \dots \end{aligned}$$

Since $H_{bs}^*(U_1)$ and $H_{bs}^*(U_2)$ vanish we conclude that $H_{bs}^l(U) \cong H_{bs}^{l-1}(V)$ for every $l \geq 1$, as desired. \square

3.3. Complexes. A *double complex* is a collection of Abelian groups $\{C^{p,q}\}_{p,q \geq 0}$ with two maps

$$d : \bigoplus_{p,q \geq 0} C^{p,q} \rightarrow \bigoplus_{p,q \geq 0} C^{p,q+1}, \quad \delta : \bigoplus_{p,q \geq 0} C^{p,q} \rightarrow \bigoplus_{p,q \geq 0} C^{p+1,q},$$

defined by the restrictions

$$d|_{C^{p,q}} = d_{p,q} : C^{p,q} \rightarrow C^{p,q+1}, \quad \delta|_{C^{p,q}} = \delta_{p,q} : C^{p,q} \rightarrow C^{p+1,q},$$

which are differentials and commute:

$$\delta^2 = d^2 = \delta d - d \delta = 0.$$

We say that the *degree* of $C^{p,q}$ is $p + q$ and define the *total complex* of C by $C^r := \bigoplus_{p+q=r} C^{p,q}$ and

$$D : \bigoplus_{r \geq 0} C^r \rightarrow \bigoplus_{r \geq 0} C^{r+1},$$

defined by the restrictions $D|_{C^r} = D_r : C^r \rightarrow C^{r+1}$, which in turn is defined by $D_r|_{C^{p,q}} = (-1)^q \delta_{p,q} + d_{p,q}$. One can verify that $D^2 = 0$. The total cohomologies of the double complex are $H_C^r := \ker D_r / \text{Im } D_{r-1}$.

Lemma 3.4. *If δ is exact at all groups of degree r , then any $\alpha \in H_C^r$ has a representative $a \in C^{0,r}$.*

Proof. Let $\alpha \in C^r$ for which $D\alpha = 0$. We will find $\beta \in C^{r-1}$ such that $\alpha + D\beta \in C^{0,r}$. Assume that

$$\alpha = \sum_{p+q=r, p \leq l} \alpha^{p,q} \in \bigoplus_{p+q=r, p \leq l} C^{p,q}, \quad (3.3)$$

where $\alpha^{p,q} \in C^{p,q}$, $\alpha^{l,r-l} \neq 0$, and $l > 0$. We will show that there is $\beta \in C^{r-1}$ that satisfies $\alpha + D\beta \in \bigoplus_{p+q=r, p \leq l-1} C^{p,q}$. Iterating this process yields the desired result.

Since $D\alpha = 0$ and l is the maximal index in the right-most term in Equation (3.3), we deduce that $\delta\alpha^{l,r-l} = 0$. Since δ is exact, there is $\beta \in C^{l-1,r-l}$ that satisfies $(-1)^{r-l}\delta\beta + \alpha^{l,r-l} = 0$. Therefore

$$\alpha + D\beta \in \bigoplus_{p+q=r, p \leq l-1} C^{p,q}.$$

□

Remark 3.5. The Proof of Lemma 3.4 is valid as soon as $D\alpha \in C^{0,r+1}$.

Define $C^{-1,q} = \ker \delta_{0,q}$. This construction has the following meaning: one can extend C to a double complex with the new cells $C^{-1,q}$. Note that the image of the restriction $D|_{C^{-1,q}} = d|_{C^{-1,q}}$ lies in $C^{-1,q+1}$, and hence $C^{-1,q}$ is a complex. We denote its cohomologies by $H_{C,d}$. Note that there is an inclusion map $C^{-1,r} \xrightarrow{i} C^r$ which induces a map $H_{C,d}^r \xrightarrow{i} H_C^r$.

Corollary 3.6. *If δ is exact then the map $H_{C,d}^r \xrightarrow{i} H_C^r$ is an isomorphism.*

Proof. By Lemma 3.4 the map $H_{C,d}^r \rightarrow H_C^r$ is onto. We will show that this map is one to one. Assume $[\alpha^{0,r}] \in H_{C,d}^r$ vanishes in H_C^r ; that is, there exists $\beta \in C^{r-1}$ such that $D\beta = \alpha^{0,r}$. By Lemma 3.4 and Remark 3.5 there exists $\gamma \in C^{r-2}$ such that $\beta^{0,r-1} = D\gamma + \beta \in C^{0,r-1}$. Thus, $\alpha^{0,r} = D\beta = D\beta^{0,r-1} = d\beta^{0,r-1}$. Since $D\beta^{0,r-1} = \alpha^{0,r-1}$, one has $\delta\beta^{0,r-1} = 0$, and thus $\alpha^{0,r-1}$ is trivial in $H_{C,d}^r$. □

3.4. The Čech-De Rham double complex. We will start this section by defining the Čech-De Rham double complex. Let \mathfrak{U} be an open cover of \mathbb{R}^n that satisfies the conditions of Theorem 1.4. Choose an arbitrary order on the set \mathfrak{U} .

Consider the following double complex:

$$\mathcal{A}^{p,q} = C^p(\mathfrak{U}, \Omega_{bs}^q) := \bigoplus_{J \subseteq \mathfrak{U}, \#J=p+1} \Omega_{bs}^q(U_J),$$

where $U_J := \bigcap_{U \in J} U$. We think of this direct sum as a subset of the direct product, and write its elements in coordinate form.

The differential $d = d_{p,q} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$ is the one defined on forms, and the differential

$$\begin{aligned} \delta = \delta_{p,q} : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p+1,q} \\ (\omega_J)_{\#J=p+1} &\mapsto (\omega'_{J'})_{\#J'=p+2} \end{aligned}$$

is the one defined by

$$\omega_J \in \Omega_{bs}^q(U_J), \quad \omega'_{J'} := \sum_{U \in J'} (-1)^{[U:J']} \omega_{J'-U} \in \Omega_{bs}^q(U_{J'}),$$

where $[U : J]$ is the index of U in J by the order induced from \mathfrak{U} ; it is 0 if U is the smallest element in J and p if it is the largest. Because only finitely many ω_J are nonzero and they all have bounded support, every $\omega'_{J'}$ vanishes outside a bounded set. Since \mathfrak{U} is locally finite, only finitely many $U_{J'}$ -s intersect any bounded set, and hence only finitely many $\omega'_{J'}$ -s are nonzero. $\mathcal{A}^{p,q}$ is the Čech-De Rham double complex. One can verify that $\delta^2 = 0$ and that δ and d commute.

One property of the Čech-De Rham double complex is that δ is exact.

Theorem 3.7. *The differential δ is exact.*

Proof. For every collection of sets J and sets $U \in J, V \notin J$ denote $J + V := J \cup \{V\}$ and $J - U := J \setminus \{U\}$. Choose a partition of unity $\{\rho_U\}_{U \in \mathfrak{U}}$. Let

$$\omega = (\omega_J)_{\#J=p+1} \in C^p(\mathfrak{U}, \Omega_{bs}^q)$$

such that only finitely many ω_J -s are nonzero. As in [2, Prop 8.5] we define

$$\begin{aligned} T : C^p(\mathfrak{U}, \Omega_{bs}^q) &\rightarrow C^{p-1}(\mathfrak{U}, \Omega_{bs}^q) \\ \omega &\mapsto (\omega'_J)_{\#J=p} \in C^{p-1}(\mathfrak{U}, \Omega_{bs}^q), \end{aligned}$$

where

$$\omega'_J := \sum_{V \notin J} (-1)^{[V:J+V]} \rho_V \omega_{J+V}.$$

Because \mathfrak{U} is locally finite, only finitely many ω'_J -s are nonzero.

Note that

$$\begin{aligned}
\delta T\omega &= \delta \left(\sum_{V \in \mathfrak{U} \setminus J} (-1)^{[V:J+V]} \rho_V \omega_{J+V} \right)_{\#J=p} \\
&= \left(\sum_{U \in J} (-1)^{[U:J]} \sum_{V \notin J-U} (-1)^{[V:J-U+V]} \rho_V \omega_{J-U+V} \right)_{\#J=p+1} \\
&= \left(\sum_{V=U \in J} (-1)^{[U:J]} (-1)^{[V:J]} \rho_V \omega_J \right)_{\#J=p+1} \\
&\quad + \left(\sum_{U \in J} (-1)^{[U:J]} \sum_{V \notin J} (-1)^{[V:J-U+V]} \rho_V \omega_{J-U+V} \right)_{\#J=p+1} \\
&= \left(\sum_{V \in J} \rho_V \omega_J \right)_{\#J=p+1} + \left(\sum_{V \notin J} \rho_V \omega_J \right)_{\#J=p+1} \\
&\quad - \left(\sum_{V \notin J} \rho_V (-1)^{[V:J+V]} \sum_{U \in J+V} (-1)^{[U:J+V]} \omega_{J-U+V} \right)_{\#J=p+1} \\
&= \omega - T\delta\omega
\end{aligned}$$

Therefore, if $\omega \in \ker \delta$ then $\omega = \delta T\omega$, and hence δ is exact. \square

Note also that $\ker \delta_{0,r}$ represents forms on \mathfrak{U} -elements that agree on pairwise intersections, and hence $\ker \delta_{0,r} \cong \Omega_{bs}^r(\mathbb{R}^n)$. From Corollary 3.6 we deduce that $H_{\mathcal{A}}^r \cong H_{bs}^r(\mathbb{R}^n)$.

Define the following double complex :

$$\mathcal{B}^{p,q} := \bigoplus_{J \subseteq \mathfrak{U}, \#J=p+1} \Omega_{bs}^q(\text{conv} U_J),$$

and define d, δ , and D as for the double complex \mathcal{A} . Denote the direct sum of the restriction transformations by $\text{res} : \mathcal{B}^{p,q} \rightarrow \mathcal{A}^{p,q}$. Since res commutes with d, δ , and D it define a map $\text{res}_* : H_{\mathcal{B}}^r \rightarrow H_{\mathcal{A}}^r$.

Proposition 3.8. *The map res_* is onto.*

Proof. Let $\alpha \in H_{\mathcal{A}}^r$. Since δ is exact and by Corollary 3.6, we have $H_{\mathcal{A}}^r \cong H_{d,A}^r \cong H_{bs}^r(\mathbb{R}^d)$, and therefore the class α corresponds to a class $[\omega] \in H_{bs}^r(\mathbb{R}^d)$. Choosing $\beta := (\omega|_{\text{conv} U})_{U \in \mathfrak{U}} \in \mathcal{B}^{0,r}$ we get $\alpha = [\text{res}\beta]$ and $\delta\beta = d\beta = D\beta = 0$. In particular, $\alpha \in \text{Im } \text{res}_*$. \square

Proposition 3.9. *At the groups $\mathcal{B}^{p,q}$ of degree n the differential d is exact.*

Proof. It is enough to show that if $p + q = n$ and $J \subseteq \mathfrak{U}$ is of size $p + 1$, then $H_{bs}^q(\text{conv}U_J) = 0$. By Theorem 3.3, the only nontrivial cohomology of $\text{conv}U_J$ may be at rank $\text{invdim conv}U_J$, and by the assumptions of Theorem 1.4 $\text{invdim conv}U_J \leq n - (p + 1) = q - 1$. Thus the q boundedly supported cohomology of $\text{conv}U_J$ is trivial, as desired. \square

Proof of Theorem 1.4. Since $\deg \mathbb{R}^n = n$ it follows that $H_{bs}^n(\mathbb{R}^n) \cong \mathbb{R} \not\cong 0$. Since $H_{bs}^n(\mathbb{R}^n) \cong H_{\mathcal{A}}^n$ we deduce that $H_{\mathcal{A}}^n \not\cong 0$. Since res_* is onto it follows that $H_{\mathcal{B}}^n \not\cong 0$. By Lemma 3.4 and the exactness of d at the groups $\mathcal{B}^{p,q}$ of degree n , we get that $\mathcal{B}^{n,0} \not\cong 0$. Thus, for some $J \subseteq \mathfrak{U}$ of size $n + 1$ the set U_J is nonempty. \square

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